

THE FUNDAMENTAL GROUP OF AFFINE CURVES IN POSITIVE CHARACTERISTIC

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ABSTRACT. It is shown that the commutator subgroup of the fundamental group of a smooth irreducible affine curve over an uncountable algebraically closed field k of positive characteristic is a profinite free group of rank equal to the cardinality of k .

1. INTRODUCTION

The algebraic (étale) fundamental group of an affine curve over an algebraically closed field k of positive characteristic has a complicated structure. It is an infinitely generated profinite group, in fact the rank of this group is same as the cardinality of k . Though the situation when k has characteristic zero is simpler to understand. The fundamental group of smooth curves over algebraically closed characteristic zero field is just the profinite completion of the topological fundamental group ([SGA1, XIII, Corollary 2.12, page 392]). In positive characteristic as well, Grothendieck gave a description of the prime-to- p quotient of the fundamental group of smooth curves which is in fact analogous to the characteristic zero case. From now on, we shall assume that the characteristic of the base field k is $p > 0$. Consider the following exact sequence for the fundamental group of a smooth affine curve C .

$$1 \rightarrow \pi_1^c(C) \rightarrow \pi_1(C) \rightarrow \pi_1^{ab}(C) \rightarrow 1$$

where $\pi_1^c(C)$ and $\pi_1^{ab}(C)$ are the commutator subgroup and the abelianization of the fundamental group $\pi_1(C)$ of C respectively. In [Ku2], a description of $\pi_1^{ab}(C)$ was given [Ku2, Corollary 3.5] and it was also shown that $\pi_1^c(C)$ is a free profinite group of countable rank if k is countable [Ku2, Theorem 1.2]. In fact some more exact sequences with free profinite kernel like the above was also observed [Ku2, Theorem 7.1]. Later using some what similar ideas and some profinite group theory Pacheco, Stevenson and Zalesskii found a condition for a closed normal subgroups of $\pi_1(C)$ to be profinite free of countable rank [PSZ]. The main result of this paper generalizes [Ku2, Theorem 1.2] to uncountable fields.

Theorem 1.1. *Let C be a smooth affine curve over an algebraically closed field k (possibly uncountable) of characteristic p then $\pi_1^c(C)$ is a free profinite group of rank $\text{card}(k)$.*

The primary reason for the algebraic fundamental group of affine curve to be incomprehensible is the wild ramification. More precisely there are many p -cyclic Artin-Schrier covers. In fact there are moduli of such covers (see [Pri]). This also suggests that the fundamental group of affine curves in positive characteristic contains much more information about the curve than what is the case in characteristic zero. In fact Harbater and Tamagawa has conjectured that the fundamental group

of smooth affine curves over algebraically closed field of characteristic p should determine the curve completely (as a scheme) and in particular one should be able to recover the base field. Harbater and Tamagawa has shown some positive results supporting the conjecture. See [Ku1, Section 3.4], [Ha5], [Ta1] and [Ta2] for more details.

The above theorem on the commutator subgroup can also be interpreted as an analogue of the Shafarevich's conjecture for global fields. Recall that the Shafarevich conjecture says that the commutator subgroup of the absolute Galois group of the rational numbers \mathbb{Q} is a profinite free group of countable rank. David Harbater [Ha3], Florian Pop [Pop] and later Dan Haran and Moshe Jarden [HJ] have shown, using different patching methods, that the absolute Galois group of the function field of a curve over an algebraically closed field is profinite free of the rank same as the cardinality of the base field. See [Ha4] for more details on these kind of problems.

Though the whole fundamental group is not understood well, a necessary and sufficient condition for a finite group to be a quotient of the fundamental group of a smooth affine curve was conjectured by Abhyankar. It was proved by Raynaud for the affine line [Ra1] and by Harbater in general [Ha1].

Theorem 1.2. (*Harbater, Raynaud*) *Let C be a smooth affine curve of genus g over an algebraically closed field of characteristic p . Let D be the smooth compactification of C and $\text{card}(D \setminus C) = n + 1$. Then a finite group G is a quotient of $\pi_1(C)$ if and only if $G/p(G)$ is generated by $2g + n$ elements, where $p(G)$ is the quasi- p subgroup of G .*

Section 2 consists of definition and results on profinite groups. This section also reduces theorem 1.1 to solving certain embedding problems. The last section consists of solutions to these embedding problems. Thanks are due to David Harbater for some useful discussions regarding the uncountable case.

2. PROFINITE GROUP THEORY

Notations and contents of this section are inspired from [RZ] and [FJ]. For a finite group G and a prime number p , let $p(G)$ denote the subgroup of G generated by all the p -Sylow subgroups. $p(G)$ is called the *quasi- p* subgroup of G . If $G = p(G)$ then G is called a quasi- p group.

A family of finite groups \mathcal{C} is said to be *almost full* if it satisfies the following conditions:

- (1) A nontrivial group is in \mathcal{C} .
- (2) If G is in \mathcal{C} then every subgroup of G is in \mathcal{C} .
- (3) If G is in \mathcal{C} then every homomorphic image of G is in \mathcal{C} .

Moreover \mathcal{C} is called a *full* family if it is closed under extensions. i.e., if G_1 and G_3 are in \mathcal{C} and there is a short exact sequence

$$1 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 1$$

then G_2 is in \mathcal{C} .

Example. The family of all finite groups is full. For a prime number p , the family of all p -groups is full.

Let \mathcal{C} be an almost full family of finite groups. A *pro- \mathcal{C}* group is a profinite group whose finite quotients lie in \mathcal{C} . Equivalently, its an inverse limit of an inverse system of groups contained in \mathcal{C} . If \mathcal{C} is the family of p -groups then *pro- \mathcal{C}* groups are also called *pro- p* groups.

Let m be an infinite cardinal or a positive integer. A subset I of a profinite group Π is called a *generating set* if the smallest closed subgroup of Π containing I is Π itself. A generating set I is said to be *converging to 1* if every open normal subgroup of Π contains all but finitely many elements of I . The *rank* of Π is the infimum of the cardinalities of each generating set of Π converging to 1.

A profinite group Π is called a *free pro- \mathcal{C}* group of rank m if Π is isomorphic to the pro- \mathcal{C} completion of an abstract free group F_I , where I is a set of cardinality m . In other words Π is the inverse limit of the inverse system obtained by taking quotients of F_I by open normal subgroups K which contain all but finitely many elements of I and $F_m/K \in \mathcal{C}$. The image of I under the natural map $F_I \rightarrow \hat{F}_I = \Pi$ is a generating set converging to 1. When \mathcal{C} is the family of all finite groups then free pro- \mathcal{C} groups are same as free profinite groups.

For a group Π and a finite group S , let $R_S(\Pi)$ denote the cardinality of the maximal ordinal m such that there exist a surjection from Π to product of m copies of S . The intersection of proper maximal subgroups of Π is called the *Frattini subgroup* of Π and is denoted by $M(\Pi)$.

An *embedding problem* consists of surjections $\phi: \Pi \twoheadrightarrow G$ and $\alpha: \Gamma \twoheadrightarrow G$

$$\begin{array}{ccccccc}
 & & & & \Pi & & \\
 & & & \swarrow \psi & \downarrow \phi & & \\
 1 & \longrightarrow & H & \longrightarrow & \Gamma & \xrightarrow{\alpha} & G \longrightarrow 1 \\
 & & & & & & \downarrow \\
 & & & & & & 1
 \end{array}$$

where G , Γ and π are groups and $H = \ker(\alpha)$. It is also sometimes called an embedding problem for Π . It is said to have a *weak solution* if there exists a group homomorphism ψ which makes the diagram commutative, i.e., $\alpha \circ \psi = \phi$. Moreover, if ψ is an epimorphism then it is said to have a *proper solution*. It is said to be a finite embedding problem if Γ is finite. An embedding problem is said to be a *split* if there exists a group homomorphism from G to Γ which is a right inverse of α .

Let \mathcal{C} be an almost full family. A profinite group Π of rank m is called *\mathcal{C} -homogeneous* if every embedding problem of the following type has a solution:

$$\begin{array}{ccccccc}
 & & & & \Pi & & \\
 & & & \swarrow & \downarrow & & \\
 1 & \longrightarrow & \tilde{H} & \longrightarrow & \tilde{\Gamma} & \longrightarrow & \tilde{G} \longrightarrow 1 \\
 & & & & & & \downarrow \\
 & & & & & & 1
 \end{array}$$

Here $\tilde{\Gamma}$ and \tilde{G} can be any pro- \mathcal{C} groups with $\text{rank}(\tilde{\Gamma}) \leq m$ and $\text{rank}(\tilde{G}) < m$. Moreover, $\tilde{H} \in \mathcal{C}$ is a minimal normal subgroup of $\tilde{\Gamma}$ and is contained in $M(\tilde{\Gamma})$. If \mathcal{C} is the class of all finite groups then Π is called *homogeneous*.

Let m denote an infinite cardinal and \mathcal{C} be an almost full family. The following is an easy generalization of [FJ, Lemma 24.17]. The proof is exactly the same but is reproduced here for completion.

Lemma 2.1. *Let Π be a profinite group such that every nontrivial finite embedding problem $\phi : \pi \twoheadrightarrow G, \alpha : \Gamma \twoheadrightarrow G$ with Γ in \mathcal{C} and $H = \ker(\alpha)$ a minimal normal subgroup of Γ has m solutions. Then the following embedding problem has a solution.*

(2.1)

$$\begin{array}{ccccccc} & & & \Pi & & & \\ & & & \downarrow & & & \\ 1 & \longrightarrow & \tilde{H} & \longrightarrow & \tilde{\Gamma} & \longrightarrow & \tilde{G} \longrightarrow 1 \\ & & & & \swarrow & & \downarrow \\ & & & & & & 1 \end{array}$$

Here $\tilde{H} \in \mathcal{C}$ is a minimal normal subgroup of $\tilde{\Gamma}$. $\tilde{\Gamma}$ and \tilde{G} are pro- \mathcal{C} groups with $\text{rank}(\tilde{\Gamma}) \leq m$ and $\text{rank}(\tilde{G}) < m$.

Moreover, if the solutions to only those embedding problems with $\tilde{H} \subset M(\tilde{\Gamma})$ is desired then the hypothesis can be weakened to the existence of m solutions to finite embedding problems in which $H = \ker(\alpha)$ is contained in $M(\Gamma)$.

Proof. Consider the embedding problem (2.1). Since \tilde{H} is finite, there exist an open normal subgroup N of $\tilde{\Gamma}$ such that $N \cap \tilde{H} = \{1\}$. Quotienting by N , we get a finite embedding problem

$$\begin{array}{ccccccc} & & & \Pi & & & \\ & & & \downarrow & & & \\ 1 & \longrightarrow & H & \longrightarrow & \Gamma & \xrightarrow{\alpha_0} & G \longrightarrow 1 \\ & & & & \swarrow & & \downarrow \\ & & & & & & 1 \end{array}$$

where $\Gamma = \tilde{\Gamma}/N$, $G = \tilde{G}/\alpha N$, the subgroup H of Γ is isomorphic to \tilde{H} and Γ is in \mathcal{C} . Moreover if we assume that $\tilde{H} \subset M(\tilde{\Gamma})$ then $H \subset M(\Gamma)$. The reason being, every maximal normal subgroup of Γ is a quotient of a maximal normal subgroup of $\tilde{\Gamma}$ containing N . The rest of the proof is same as that of [FJ, Lemma 24.17]. We have the following scenario:

$$\begin{array}{ccccc} \Pi & & & & \\ & \searrow \phi & & & \\ \tilde{H} & \hookrightarrow & \tilde{\Gamma} & \xrightarrow{\alpha} & \tilde{G} \\ \parallel & & \downarrow & & \downarrow \\ H & \hookrightarrow & \Gamma & \xrightarrow{\alpha_0} & G \end{array}$$

By assumption there exist $\beta : \Pi \twoheadrightarrow \Gamma$ which makes the above diagram commutative. In fact there are m choices for β . If $\ker(\phi) \subset \ker(\beta)$ then β factors through \tilde{G} . By [FJ, Lemma 24.13], there are at most $\text{rank}(\tilde{G}) < m$ surjections from \tilde{G} to Γ . Hence

we can choose β so that $\ker \beta$ does not contain $\ker \phi$. Since $\tilde{\Gamma}$ is the fiber product of \tilde{G} and Γ over G , the maps β and ϕ induce a map $\gamma : \Pi \rightarrow \tilde{\Gamma}$ so that the following diagram commutes:

$$\begin{array}{ccccc} \Pi & & & & \\ & \searrow \gamma & & \searrow \phi & \\ & \tilde{\Gamma} & \xrightarrow{\alpha} & \tilde{G} & \\ & \downarrow & & \downarrow & \\ & \Gamma & \xrightarrow{\alpha_0} & G & \end{array}$$

By [FJ, Lemma 23.10] there exist a group G' which fits in the following diagram:

$$\begin{array}{ccccc} \text{Im}(\gamma) & \xrightarrow{\alpha} & \tilde{G} & & \\ \downarrow & & \downarrow \zeta' & \searrow & \\ \Gamma & \xrightarrow{\alpha'_0} & G' & \xrightarrow{\theta} & G \\ & \searrow \alpha_0 & & & \end{array}$$

the maps from $\text{Im}(\gamma)$ are the restriction of maps from $\tilde{\Gamma}$ and $\text{Im}(\gamma)$ is the fiber product of \tilde{G} and Γ over G' . Since $H = \ker(\alpha_0)$ is a minimal normal subgroup of Γ , one of θ or α'_0 is an isomorphism. If α'_0 were an isomorphism then $\beta = \alpha'_0{}^{-1} \circ \zeta' \circ \phi$ contradicting $\ker(\phi)$ is not a subset of $\ker(\beta)$. Hence θ is an isomorphism. So again by [FJ, Lemma 23.10], $\text{Im}(\gamma) = \tilde{\Gamma}$ solving the embedding problem (2.1). \square

Theorem 2.2. *Let Π be profinite group of rank m . Suppose:*

- (1) Π is projective.
- (2) Every nontrivial finite split embedding problem

(2.2)

$$\begin{array}{ccccccc} & & & & \Pi & & \\ & & & & \downarrow & & \\ 1 & \longrightarrow & H & \longrightarrow & \Gamma & \longrightarrow & G \longrightarrow 1 \\ & & & & \swarrow & & \downarrow \\ & & & & & & 1 \end{array}$$

with H minimal normal subgroup of Γ and $H \subset M(\Gamma)$ has m solutions.

Then Π is homogeneous. Moreover, if $R_S(\Pi) = m$ for every finite simple group S then Π is a profinite free group.

Proof. First of all, let us observe that (1) and (2) allows us to assume that every finite nontrivial (not necessarily split) embedding problem (2.2) with $H \subset M(\Gamma)$ has m solutions. The proof is via induction on $|H|$. The embedding problem (2.2) has weak solution ϕ since Π is projective. Let $G' \leq \Gamma$ be the image of ϕ . G' acts on H via conjugation so we can define $\Gamma' = H \rtimes G'$ and get the following embedding

problem:

(2.3)

$$\begin{array}{ccccccc}
 & & & \Pi & & & \\
 & & & \downarrow \phi & & & \\
 & & \swarrow & & \searrow & & \\
 1 & \longrightarrow & H & \longrightarrow & \Gamma' & \longrightarrow & G' \longrightarrow 1 \\
 & & & & & & \downarrow \\
 & & & & & & 1
 \end{array}$$

Also Γ' surjects onto Γ under the homomorphism sending $(h, g) \mapsto hg$. So it is enough to find m solutions to the embedding problem (2.3). Now if H is not a minimal normal subgroup of Γ' then there exist H' proper nontrivial subgroup of H and normal in Γ' . Quotienting by H' we get the following embedding problem:

$$\begin{array}{ccccccc}
 & & & \Pi & & & \\
 & & & \downarrow \phi & & & \\
 & & \swarrow & & \searrow & & \\
 1 & \longrightarrow & H/H' & \longrightarrow & \Gamma'/H' & \longrightarrow & G' \longrightarrow 1 \\
 & & & & & & \downarrow \\
 & & & & & & 1
 \end{array}$$

which has m solutions by induction hypothesis (since $|H/H'| < |H|$). For each solution θ' to the above, the following embedding problem:

$$\begin{array}{ccccccc}
 & & & \Pi & & & \\
 & & & \downarrow \theta' & & & \\
 & & \swarrow & & \searrow & & \\
 1 & \longrightarrow & H' & \longrightarrow & \Gamma' & \longrightarrow & \Gamma'/H' \longrightarrow 1 \\
 & & & & & & \downarrow \\
 & & & & & & 1
 \end{array}$$

also has m solutions by induction hypothesis as $|H'| < |H|$. Let θ be solution to this embedding problem then it is in fact a solution to (2.3) as well. Finally if H is a minimal normal subgroup of Γ' then hypothesis (2) guarantees m solutions to (2.3). Lemma 2.1 yields Π is homogeneous. The rest of the statement follows from [RZ, Theorem 8.5.2] and [RZ, Lemma 3.5.4]. \square

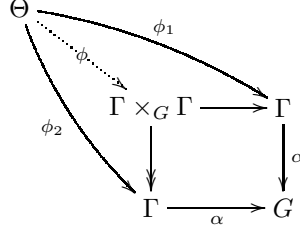
Let Γ be the semidirect product of finite groups G and H , with H a minimal normal subgroup of Γ contained in $M(\Gamma)$ and let Π be a closed normal subgroup of a profinite group Θ .

Lemma 2.3. *Suppose we have a surjection ψ from $\Theta \rightarrow G$ which restricted to Π is also a surjection.*

$$\begin{array}{ccc}
 \Pi & \hookrightarrow & \Theta \\
 & \searrow & \downarrow \psi \\
 H & \hookrightarrow & \Gamma \xrightarrow{\alpha} G
 \end{array}$$

Let ϕ_1 and ϕ_2 be distinct surjections of Θ onto Γ such that $\psi = \alpha \circ \phi_1 = \alpha \circ \phi_2$. Then restrictions of ϕ_1 and ϕ_2 to Π are distinct surjections onto Γ .

Proof. Let us first note that for $i = 1, 2$, $\phi_i(\Pi)$ contains G and is a normal subgroup of Γ and $H \subset M(\Gamma)$, so $\phi_i(\Pi) = \Gamma$. The surjections ϕ_1 and ϕ_2 onto Γ induce a map $\phi : \Theta \rightarrow \Gamma \times_G \Gamma = \{((h_1, g), (h_2, g)) | h_1, h_2 \in H, g \in G\}$.



Let K , K_1 and K_2 be the kernels of ϕ , ϕ_1 and ϕ_2 respectively. Then $K = K_1 \cap K_2$. Note that $|\Gamma \times_G \Gamma| = |G||H|^2$ hence to show ϕ is surjective it is enough to show $|\Theta/K| = |G||H|^2$. But the index $[\Theta : K] = [\Theta : K_1][K_1 : K]$ and $[\Theta : K_1] = |G||H|$, so it is enough to show $K_1/K \cong H$. As $0 = \alpha \circ \phi_1(K_1) = \alpha \circ \phi_2(K_1)$, $\phi_2(K_1) \subset \ker \alpha = H$ and is a normal subgroup of Γ . $K_1 \neq K_2$ ensures that $\phi_2(K_1)$ is a non trivial subgroup of H . Since H is a minimal normal subgroup of Γ , we conclude $\phi(K_1) \cong H$. So $K_1/K \cong H$ and hence ϕ is surjective.

Also since $H \subset M(\Gamma)$, the subgroup $H \times H = \{((h_1, 1), (h_2, 1)) | h_1, h_2 \in H\}$ of $\Gamma \times_G \Gamma$ is contained in $M(\Gamma \times_G \Gamma)$. Clearly $\phi(\Pi)$ contains G and is a normal subgroup $\Gamma \times_G \Gamma$. Hence $\phi(\Pi) = \Gamma \times_G \Gamma$. So $K_1 \cap \Pi \neq K_2 \cap \Pi$ and hence restriction of ϕ_1 and ϕ_2 to Π are distinct. \square

3. SOLUTION TO EMBEDDING PROBLEMS

A morphism of schemes, $\Phi : Y \rightarrow X$, is said to be a *cover* if Φ is finite, surjective and generically separable. For a finite group G , Φ is said to be a *G-cover* (or a *G-Galois cover*) if in addition there exists a group homomorphism $G \rightarrow \text{Aut}_X(Y)$ which acts transitively on the geometric generic fibers of Φ .

Let k be an uncountable algebraically closed field of characteristic p . Let K^{un} denote the compositum (in some fixed algebraic closure of $k(C)$) of the function fields of all Galois étale covers of C . In these notations $\pi_1(C) = \text{Gal}(K^{un}/k(C))$. For each $g \geq 0$, let

$$P_g(C) = \cap \{\pi_1(Z) : Z \rightarrow C \text{ is an étale } p\text{-cyclic cover and genus of } Z \geq g\}$$

be a sequence of increasing closed normal subgroups of $\pi_1(C)$.

Let Π be a closed normal subgroup of $\pi_1(C)$ of rank m and $g \geq 0$ be such that $\Pi \subset P_g(C)$. We shall prove the following:

Theorem 3.1. *Π is a homogeneous profinite group. Moreover if $R_S(\Pi) = m$ for every finite simple group S then Π is a free profinite group of rank m .*

Let K^b be the fixed subfield of K^{un} under the action of Π . By infinite Galois theory $\text{Gal}(K^{un}/K^b) = \Pi$. Giving a surjection from Π to a finite group G is same as giving a Galois extension $M \subset K^{un}$ of K^b with Galois group G . Since K^b is an algebraic extension of $k(C)$ and M is a finite extension of K^b , we can find a finite extension $L \subset K^b$ of $k(C)$ and $L' \subset K^{un}$ a G -Galois extension of L so that $M = K^b L'$. Let $\pi_1^L = \text{Gal}(K^{un}/k(X))$. The following figure summarizes the above situation.

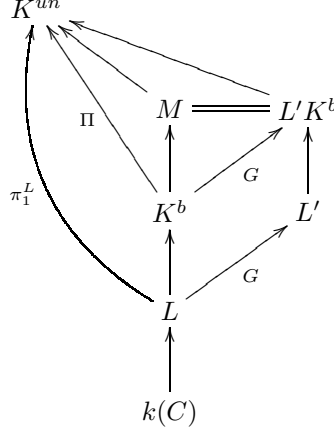


FIGURE 1.

Let D be the smooth compactification of C , X be the normalization of D in L and $\Phi'_X : X \rightarrow D$ be the normalization morphism. Then X is an abelian cover of D étale over C and its function field $k(X)$ is L . Let W_X be the normalization of X in L' and Ψ_X corresponding normalization morphism. Then Ψ_X is étale away from points lying above $D \setminus C$ and $k(W_X) = L'$. Since k is algebraically closed, $k(C)/k$ has a separating transcendence basis. By a stronger version of Noether normalization (for instance, see [EH, Corollary 16.18]), there exist a finite proper k -morphism from C to \mathbb{A}_x^1 , where x denotes the local coordinate of the affine line, which is generically separable. The branch locus of such a morphism is codimension 1, hence this morphism is étale away from finitely many points. By translation we may assume none of these points map to $x = 0$. This morphism extends to a finite proper morphism $\Theta : D \rightarrow \mathbb{P}_x^1$. Let $\Phi_X : X \rightarrow \mathbb{P}_x^1$ be the composition $\Theta \circ \Phi'_X$. Let $\{r_1, \dots, r_N\} = \Phi_X^{-1}(\{x = 0\})$, then Φ_X is étale at r_1, \dots, r_N . Also note that $\Theta^{-1}(\{x = \infty\}) = D \setminus C$.

Let k_0 be a countable algebraically closed subfield of k such that the morphisms Φ_X and Ψ_X are defined over k_0 . Let \mathcal{F} be a totally ordered set of algebraically closed subfields of k containing k_0 under inclusion such that for any k_α in \mathcal{F} , transcendence degree of k_α over $\cup\{k' \in \mathcal{F} : k' \subsetneq k_\alpha\}$ is at least countably infinite. Note that since k is uncountable, we can find such a \mathcal{F} with same cardinality as that of k .

3.1. Prime-to- p group. Assume H is a prime-to- p group and a minimal normal subgroup of Γ with $\Gamma/H = G$. Let $\Phi_Y : Y \rightarrow \mathbb{P}_y^1$ be a $\mathbb{Z}/p\mathbb{Z}$ -cover ramified only at $y = 0$ given by $Z^p - Z - y^{-r}$ where r is coprime to p and can be chosen to ensure that the genus of Y is as large as desired. Let F be the locus of $t - xy = 0$ in $\mathbb{P}_x^1 \times_k \mathbb{P}_y^1 \times_k \times_k \text{Spec}(k[[t]])$. Let $Y_F = Y \times_{\mathbb{P}_y^1} F$, where the morphism from $F \rightarrow \mathbb{P}_y^1$ is given by the composition of morphisms $F \hookrightarrow \mathbb{P}_x^1 \times_k \mathbb{P}_y^1 \times_k \text{Spec}(k[[t]]) \rightarrow \mathbb{P}_y^1$. Similarly define $X_F = X \times_{\mathbb{P}_x^1} F$. Let T be the normalization of an irreducible dominating component of the fiber product $X_F \times_F Y_F$. The situation so far can be described by the following picture:

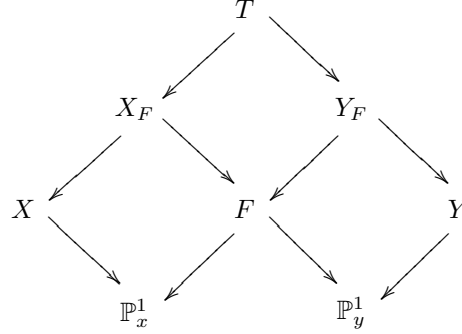


FIGURE 2.

Let $\Psi_Y : W_Y \rightarrow Y$ be an étale H -cover of Y . Note that this is possible because H is a prime-to- p group and the genus of Y can be made arbitrarily large ([SGA1, XIII, Corollary 2.12, page 392]). Y (or r above) is chosen so that the genus of Y is at least g and greater than the number of generators for H .

Proposition 3.2. *For each $k_\alpha \in \mathcal{F}$, there exist an irreducible Γ -cover $W_\alpha \rightarrow T_\alpha$ with the following properties:*

- (1) T_α is a cover of D unramified over C , the composition $W_\alpha \rightarrow T_\alpha \rightarrow D$ is unramified over C and W_α dominates W_X .
- (2) The cover $W_\alpha \rightarrow T_\alpha$ is not defined over $k' = \cup \{k_\beta \in \mathcal{F} : k_\beta \subsetneq k_\alpha\}$.
- (3) T_α is a Galois cover of D with $\Pi \subset \text{Gal}(K^{un}/k(T_\alpha))$, i.e., $k(T_\alpha) \subset K^b$.

Proof. By [Ku2, Proposition 6.4] there exist an irreducible normal Γ -cover $W \rightarrow T$ of $k[[t]]$ -schemes such that over the generic point $\text{Spec}(k((t)))$, $W^g \rightarrow T^g$ is ramified only over the points of T^g lying above $x = \infty$.

Now this cover can be specialized to covers of k -schemes using [Ku2, Proposition 6.9] to obtain (1). In fact it was shown that there exist an open subset S of the spectrum of a $k[t, t^{-1}]$ algebra such that fiber over every point in S leads to a cover with desired ramification properties and Galois groups. So any fiber over a k_α -point will satisfy (1).

(2) is achieved by choosing a specialization over a k_α -point of S which is not a k' -point. More specifically, note that in the above the morphism $W^g \rightarrow \mathbb{P}_x^1 \times \text{Spec}(k((t)))$ of $k((t))$ -schemes factors through $\lambda : Y_F \times_{k[[t]]} \text{Spec}(k((t))) \rightarrow \mathbb{P}_x^1 \times \text{Spec}(k((t)))$. The latter covering map is locally given by $Z^p - Z - (x/t)^r$. By Artin-Schreier theory for p -cyclic extensions, the field extension given by irreducible polynomials $Z^p - Z - ax^r$ and $Z^p - Z - bx^r$ over $k(x)$ for $a, b \in k$ are equal if and only if $a/b \in \mathbb{F}_p$. Since k_α has infinite transcendence degree over k' there exist $a_\alpha \in k_\alpha$ transcendental over k' such that $t = a_\alpha$ defines a point P_α in S which is not a k' -point. Then the cover $Y_{P_\alpha} \rightarrow \mathbb{P}_x^1$ is not defined over k' . Hence the cover W_{P_α} a cover of Y_{P_α} is not defined over k' .

For (3), first we observe that $k(T)$ is a compositum of $L_1 = k(X) \otimes_k k((t))$ and L_2 , where L_2 is the function field of a dominating irreducible component of

$$(Y \times_k \text{Spec}(k((t)))) \times_{\mathbb{P}_y^1 \times_k \text{Spec}(k((t)))} (D \times_k \text{Spec}(k((t))))$$

Here the morphism $D \times_k \text{Spec}(k((t))) \rightarrow \mathbb{P}_y^1 \times_k \text{Spec}(k((t)))$ is the composition of $D \times_k \text{Spec}(k((t))) \rightarrow \mathbb{P}_x^1 \times_k \text{Spec}(k((t)))$ with $\mathbb{P}_x^1 \times_k \text{Spec}(k((t))) \rightarrow \mathbb{P}_y^1 \times_k \text{Spec}(k((t)))$ where the later morphism is defined in local co-ordinates by sending y to t/x . Let Z be the normalization of $D \times_k \text{Spec}(k((t)))$ in L_2 .

Since T_α is a specialization of T to k , $k(T_\alpha)$ is a compositum of $k(X)$ and $k(Z_\alpha)$ where Z_α is the corresponding specialization of Z to k . The morphism from $Z \rightarrow D \times_k \text{Spec}(k((t)))$ factors through $Y \times_k \text{Spec}(k((t)))$. So genus of Z and hence of Z_α is greater than g . Also $Z_\alpha \rightarrow D$ is unramified over C because T_α dominates Z_α and the Galois extension $k(Z)/k((t))(D)$ is the compositum of the p -cyclic extension $k((t))(Y)/k((t))(t/x)$ by $k((t))(D)$. Hence $k(Z)/k((t))(D)$ is either trivial or p -cyclic and the same is true for $k(Z_\alpha)/k(D)$. If $k(Z_\alpha)/k(D)$ is trivial then $\text{Gal}(K^{un}/k(Z_\alpha)) = \pi_1(C) \supset \Pi$ by assumption. If $k(Z_\alpha)/k(D)$ is a p -cyclic extension then $\text{Gal}(K^{un}/k(Z_\alpha)) \supset P_g(C)$ as genus of Z_α is greater than g . So either way $\Pi \subset \text{Gal}(K^{un}/k(Z_\alpha))$. Π is clearly contained in $\pi_1^L = \text{Gal}(K^{un}/k(X))$ (see Figure 1), hence $\Pi \subset \text{Gal}(K^{un}/k(T_\alpha))$. \square

Theorem 3.3. *Under the above notation the following split embedding problem has $\text{card}(k) = m$ proper solutions*

$$\begin{array}{ccccccc}
 & & & \Pi & & & \\
 & & & \downarrow & & & \\
 1 & \longrightarrow & H & \longrightarrow & \Gamma & \longrightarrow & G \longrightarrow 1 \\
 & & & & \swarrow & & \downarrow \\
 & & & & & & 1
 \end{array}$$

Here H is a prime to p -group minimal normal subgroup of Γ .

Proof. By proposition 3.2, for each $k_\alpha \in \mathcal{F}$, there exist W_α and T_α such that $\text{Gal}(k_\alpha(W_\alpha)/k_\alpha(T_\alpha))$ is Γ . Moreover, since k_α and k are algebraically closed field, $\text{Gal}(k(W_\alpha)/k(T_\alpha))$ is also Γ . Also $k(T_\alpha) \subset K^b$, so $\text{Gal}(k(W_\alpha)/k(W_X)k(T_\alpha)) = H$. Since H is a prime-to- p group and there are only finitely many H -covers of W_X which are étale over the preimage of C , for all but finitely many of these α , $\text{Gal}(k(W_\alpha)K^b/k(W_X)K^b) = H$. Hence for all but finitely many of these α , $\text{Gal}(k(W_\alpha)K^b/K^b) = \Gamma$. In particular, for each of these α ,

$$(3.1) \quad k(W_\alpha)k(T_\alpha) \text{ and } K^b k(W_X) \text{ are linearly disjoint over } k(W_X)k(T_\alpha)$$

Removing these finitely many fields from \mathcal{F} , we may assume this holds for all $k_\alpha \in \mathcal{F}$.

So to complete the proof, it is enough to show that for $k_\alpha \subsetneq k_\beta \in \mathcal{F}$, $k(W_\alpha)K^b \neq k(W_\beta)K^b$ as subfields of K^{un} . Let $L_o = k_\alpha(T_\alpha)k_\beta(T_\beta)$. Note that $L_o \subset K^b$. We will first show that $k_\alpha(W_\alpha)L_o \neq k_\beta(W_\beta)L_o$. Suppose equality holds, then the extension $k_\beta(W_\beta)k_\alpha(T_\alpha)/k_\beta(T_\beta)k_\alpha(T_\alpha)$ is a base change of the extension $k_\alpha(W_\alpha)/k_\alpha(T_\alpha)$ by $k_\beta(T_\beta)$. In particular the morphism $W_\beta \rightarrow T_\beta$ is defined over k_α which contradicts Proposition 3.2. Since k and k_β are algebraically closed fields, $k(W_\alpha)L_o \neq k(W_\beta)L_o$. Moreover, both these fields are Γ -extensions of the compositum kL_o .

In fact $k(W_\alpha)L_o$ and $k(W_\beta)L_o$ are linearly disjoint over $k(W_X)L_o$. To see this, assume the contrary and let $M' = k(W_\alpha)L_o \cap k(W_\beta)L_o$. M' is a subfield of $k(W_\alpha)L_o$ which contains $k(W_X)L_o$ properly. Also being an intersection of two Galois extension of kL_o , M' is a Galois extension of kL_o . So the Galois group

$\text{Gal}(k(W_\alpha)L_o/M')$ is a normal subgroup of $\Gamma = \text{Gal}(k(W_\alpha)L_o/kL_o)$ and a proper subgroup of $H = \text{Gal}(k(W_\alpha)L_o/k(W_X)L_o)$ contradicting that H is a minimal normal subgroup of Γ .

As a consequence of equation (3.1) $k(W_\alpha)L_o$ and $k(W_\beta)L_o$ are linearly disjoint to $K^b k(W_X)$ over $k(W_X)L_o$. Since the three fields are mutually linearly disjoint over $k(W_X)L_o$, $k(W_\alpha)K^b$ and $k(W_\beta)K^b$ are linearly disjoint over $k(W_X)K^b$ and hence are distinct. \square

3.2. Quasi- p group. Now embedding problem with quasi- p kernel H contained in the Frattini subgroup will be shown to have m solutions.

Theorem 3.4. *The following split embedding problem has $\text{card}(k) = m$ proper solutions*

$$(3.2) \quad \begin{array}{ccccccc} & & & & \Pi & & \\ & & & & \downarrow & & \\ 1 & \longrightarrow & H & \longrightarrow & \Gamma & \longrightarrow & G \longrightarrow 1 \\ & & & & \swarrow & & \downarrow \\ & & & & & & 1 \end{array}$$

Here H is quasi p -group, minimal normal subgroup of Γ and is contained in $M(\Gamma)$.

Proof. Let X^0 be the normalization of C in L (see figure 1), i.e., X^0 is the open subset of X lying above C . Hence $\pi_1(X^0)$ contains Π and the surjection $\Pi \rightarrow G$ extends to $\pi_1(X^0)$. Also note that $\pi_1(X^0)$ is a subgroup of $\pi_1(C)$. By F. Pop's result ([Pop],[Ha2, Theorem 5.3.4]), the following embedding problem has m solutions

$$\begin{array}{ccccccc} & & & & \pi_1(X^0) & & \\ & & & & \downarrow & & \\ 1 & \longrightarrow & H & \longrightarrow & \Gamma & \longrightarrow & G \longrightarrow 1 \\ & & & & \swarrow & & \downarrow \\ & & & & & & 1 \end{array}$$

Let $\tilde{\theta}$ be any solution to the above embedding problem and θ be the restriction of $\tilde{\theta}$ to the normal subgroup Π . Lemma 2.3 can now be applied to conclude that θ is a solution to the embedding problem (3.2) and it has m distinct solutions. \square

Proof. (of theorem 3.1) $\pi_1(C)$ is projective so Π being a closed subgroup of $\pi_1(C)$ is also projective ([FJ, Corollary 20.14]). The result now follows from theorem 2.2, theorem 3.3, theorem 3.4 \square

Proposition 3.5. *Let S be a finite simple group and C be as in the previous theorem then $R_S(\pi_1^c(C)) = m$.*

Proof. If S is prime-to- p group then the proposition follows from Theorem 3.3 by taking $H = \Gamma = S$. If p divides $|S|$ then S is a quasi- p group. Moreover if S is also non-abelian group then the proposition follows from [Ku2, Theorem 5.3]. Finally if S is an abelian simple p -group then $S \cong \mathbb{Z}/p\mathbb{Z}$. In this case we observe that the pro- p quotient of $\pi_1(C)$ is isomorphic to the pro- p free group of rank m by [Ha2,

Theorem 5.3.4], Lemma 2.1 and [RZ, Theorem 8.5.2]. So the pro- p part of the commutator subgroup of $\pi_1(C)$ is also pro- p free of rank m [RZ, Corollary 8.9.2]. Hence $R_S(\pi_1^c(C)) = m$ for $S = \mathbb{Z}/p\mathbb{Z}$ as well. \square

Corollary 3.6. $\pi_1^c(C)$ is a profinite free group of rank m for any smooth affine curve C over an algebraically closed field k of characteristic p and cardinality m .

Proof. $\pi_1^c(C)$ is a closed normal subgroup of $\pi_1(C)$ of rank m and contains $P_0(C)$. So the result follows from theorem 3.1 and proposition 3.5. \square

Let K_{p^n} denote the intersection of all open normal subgroups of $\pi_1(C)$ so that the quotient is an abelian group of exponent at most p^n . Let $G_{p^n} = \pi_1(C)/K_{p^n}$ then $G_{p^n} = \varprojlim \text{Gal}(k(Z)/k(C))$ where $Z \rightarrow C$ is a Galois étale cover of C with Galois group $(\mathbb{Z}/p^n\mathbb{Z})^l$ for some $l \geq 1$. G_{p^n} has a description in terms of Witt rings of the coordinate ring of C . In fact $G_{p^n} \cong \text{Hom}(W_n(\mathcal{O}_C)/P(W_n(\mathcal{O}_C)), \mathbb{Z}/p^n\mathbb{Z})$ by [Ku2, Lemma 3.3]. Here $W_n(\mathcal{O}_C)$ is the ring of Witt vectors of length n and P is a group homomorphism from $W_n(\mathcal{O}_C)$ to itself given by “Frobenius - Identity” (see Section 2 of [Ku2] for details). Hence for any $n \geq 1$, we get the following exact sequence:

$$1 \rightarrow K_{p^n} \rightarrow \pi_1(C) \rightarrow \text{Hom}(W_n(\mathcal{O}_C)/P(W_n(\mathcal{O}_C)), \mathbb{Z}/p^n\mathbb{Z}) \rightarrow 1$$

Remark 3.7. K_{p^n} is a closed normal subgroup of $\pi_1(C)$ and is contained in $P_0(C)$. Proposition 3.5 is also true when $\pi_1^c(C)$ is replaced by K_{p^n} and the proof is the same. Hence K_{p^n} is also profinite free of rank m in view of theorem 3.1.

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